Computing Periodic Triangulations

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This work was initially motivated by the need for periodic Delaunay triangulations in $\mathbb{R}^3$ in numerous domains, including astronomy, material engineering, biomedical computing, fluid dynamics, etc (see [3] for references). Periodic triangulations in the hyperbolic plane $\mathbb{H}^2$ also arise in very diverse fields, such as physics, solid modeling, and neuromathematics (see [1]).

Periodic triangulations can be seen as triangulations of a manifold that is the quotient of the space ($\mathbb{R}^3$ or $\mathbb{H}^2$ in the abovementioned two cases) under the action of some discrete group of isometries (some crystallographic group for the Euclidean case, some Fuchsian group for hyperbolic case).

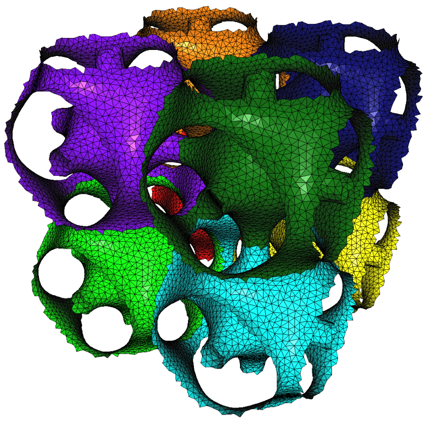

FIG. 1. Mesh of a 3D triply-periodic volume *(work with Aymeric Pellé).*

Delaunay triangulations in the Euclidean space $\mathbb{R}^d$ have been extensively studied. They are an essential tool for a large number of fields. A classic incremental algorithm consists in inserting points one after another; for each new point, the region of cells in conflict is computed, then it is partioned by new cells having the new point as vertex. This algorithm yields a very efficient implementation, publicly available in the CGAL library (`www.cgal.org`). The Delaunay complex in $\mathbb{H}^d$ can be derived from the Delaunay triangulation in $\mathbb{R}^d$ [2].

Much less is known for the case of quotient manifolds. We studied cycles in the graph of edges of the projection of the Delaunay triangulation in $\mathbb{R}^3$ or $\mathbb{H}^2$ onto the studied quotient. We studied covering spaces that allow to avoid cycles of length 1 and 2, thus guaranteeing that the projection of the Delaunay triangulation onto the covering space is a simplicial complex, which allows us to extend the algorithm described above [1, 3].

While the general scheme is similar for Euclidean and hyperbolic manifolds, the case of hyperbolic manifolds is more tricky, if only because hyperbolic translations do not commute. We could consider the case of general closed Euclidean dD manifolds, while our study in the hyperbolic case is currently limited to the Bolza surface. In the specific case of the 3D flat torus, the work led to an implementation in CGAL, and a collaboration with astrophysicists (see e.g., `http://www.computational-geometry.org/SoCG-videos/socg12video/`). Further work includes the computation of 3D Euclidean periodic meshes (Fig. 1) and plane hyperbolic meshes (Fig.2).

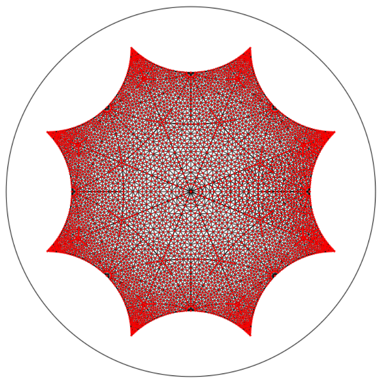

FIG. 2. Periodic mesh in the hyperbolic plane *(work with Mathieu Schmitt).*

More references can be found at `http://www.loria.fr/~teillaud/other-geometries/`

[1] M. Bogdanov and M. Teillaud. Delaunay triangulations and cycles on closed hyperbolic surface. Research Report INRIA No 8434 (2013). `http://hal.inria.fr/hal-00921157`

[2] M. Bogdanov, O. Devillers, and M. Teillaud. Hyperbolic Delaunay complexes and Voronoi diagrams made practical. Proc. SoCG, pp. 67-76, 2013. `http://hal.inria.fr/hal-00833760`

[3] M. Caroli and M. Teillaud. Computing 3D Periodic Triangulations. Proc. ESA. LNCS 5757, pp. 59–70 (2009). doi: 10.1007/978-3-642-04128-0_6. - See also SoCG'08 video at `http://www.computational-geometry.org/SoCG-videos/socg08video/`

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